烟台理工学院
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机器人学

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二零二一年八月二十日

刚体的加速度

- 速度矢量 V_Q 的微分

$$^{B}A_Q = \frac{d}{dt}\,{}^{B}V_Q = \lim_{\Delta t \to 0} \frac{{}^{B}V_Q(t+\Delta t) - {}^{B}V_Q(t)}{\Delta t}$$

速度矢量 $^{B}V_Q$ 相对于坐标系 {B} 的微分

刚体的加速度

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速度矢量 $^{B}V_Q$ 相对于坐标系 {B} 的微分

$$^{A}(^{B}A_Q) = {}^{A}(\frac{d}{dt}\,^{B}V_Q)$$

在坐标系 {A} 的表达

刚体的加速度

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速度矢量 $^{B}V_Q$ 相对于坐标系 {B} 的微分

$$^{A}({}^{B}A_Q) = {}^{A}\left(\frac{d}{dt}\,{}^{B}V_Q\right)$$

在坐标系 {A} 的表达

$$= {}^{A}_{B}R\,{}^{B}({}^{B}A_Q) = {}^{A}_{B}R\,{}^{B}A_Q$$ 两个坐标系相同时的表达

刚体的加速度

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速度矢量 ${}^{B}V_Q$ 相对于坐标系 {B} 的微分

$$ {}^{A}({}^{B}A_Q) = {}^{A}(\frac{d}{dt}\,{}^{B}V_Q) $$

在坐标系 {A} 的表达

$$ = {}^{A}_{B}R\,{}^{B}({}^{B}A_Q) = {}^{A}_{B}R\,{}^{B}A_Q $$

两个坐标系相同时的表达

$$ a_C = {}^{U}A_{C\ ORG} $$

坐标系 {C} 的原点相对于世界参考坐标系的加速度

刚体的加速度

- 角加速度矢量 ${}^{A}\dot{\Omega}_{B}$

$$ {}^{A}\dot{\Omega}_{B} = \frac{d}{dt}\,{}^{A}\Omega_{B} = \lim_{\Delta t \to 0} \frac{{}^{A}\Omega_{B}(t+\Delta t) - {}^{A}\Omega_{B}(t)}{\Delta t} $$

坐标系 {B} 相对于坐标系 {A} 的角速度的微分

刚体的加速度

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$$ {}^{A}\dot{\Omega}_{B} = \frac{d}{dt}\,{}^{A}\Omega_{B} = \lim_{\Delta t \to 0} \frac{{}^{A}\Omega_{B}(t+\Delta t) - {}^{A}\Omega_{B}(t)}{\Delta t} $$

坐标系{B}相对于坐标系{A}的角速度的微分

${}^{C}({}^{A}\dot{\Omega}_{B})$

在坐标系{C}中的表达

刚体的加速度

- 角加速度矢量 ${}^{A}\dot{\Omega}_{B}$

$$ {}^{A}\dot{\Omega}_{B} = \frac{d}{dt}\,{}^{A}\Omega_{B} = \lim_{\Delta t \to 0} \frac{{}^{A}\Omega_{B}(t+\Delta t) - {}^{A}\Omega_{B}(t)}{\Delta t} $$

坐标系 {B} 相对于坐标系 {A} 的角速度的微分

$$ {}^{C}({}^{A}\dot{\Omega}_{B}) $$

在坐标系 {C} 中的表达

$$ \dot{\omega}_{C} = {}^{U}\dot{\Omega}_{C} $$

坐标系 {C} 相对于世界参考坐标系 {U} 的角加速度

刚体的加速度

➢ 角加速度

$$^{A}\Omega_{C} = {}^{A}\Omega_{B} + {}^{A}_{B}R\,{}^{B}\Omega_{C}$$

刚体的加速度

- 角加速度

$$ {}^{A}\Omega_{C} = {}^{A}\Omega_{B} + {}^{A}_{B}R\, {}^{B}\Omega_{C} $$

$$ \Big\downarrow \text{diff.} $$

$$ {}^{A}\dot{\Omega}_{C} = {}^{A}\dot{\Omega}_{B} + \frac{d}{dt}\, {}^{A}_{B}R\, {}^{B}\Omega_{C} $$

$$ = {}^{A}\dot{\Omega}_{B} + {}^{A}_{B}R\, {}^{B}\dot{\Omega}_{C} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\, {}^{B}\Omega_{C} $$

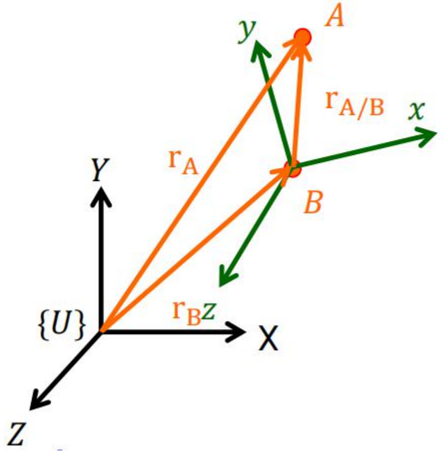
刚体的运动

- 动态运动

根据第5章的描述

$$\overrightarrow{v_A} = \overrightarrow{v_B} + \overrightarrow{v_{rel}} + \overrightarrow{\omega} \times \overrightarrow{r_{A/B}}$$

$$\overrightarrow{v_A} = (\dot{x}_B\hat{I} + \dot{y}_B\hat{J}) + (\dot{x}_{A/B}\hat{\imath} + \dot{y}_{A/B}\hat{\jmath}) + \overrightarrow{\omega} \times (x_{A/B}\hat{I} + y_{A/B}\hat{J})$$



刚体的运动

➢ 动态运动

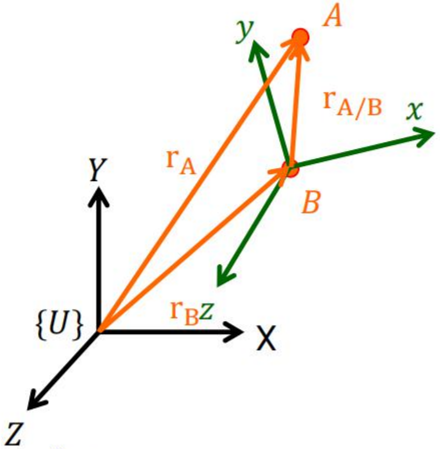
根据第5章的描述

$$\overrightarrow{v_A} = \overrightarrow{v_B} + \overrightarrow{v_{rel}} + \vec{\omega} \times \overrightarrow{r_{A/B}}$$

$$\overrightarrow{v_A} = (\dot{x}_B \hat{I} + \dot{y}_B \hat{J}) + (\dot{x}_{A/B}\hat{\imath} + \dot{y}_{A/B}\hat{\jmath}) + \vec{\omega} \times (x_{A/B}\hat{I} + y_{A/B}\hat{J})$$

↓ diff.

$$\begin{aligned}\overrightarrow{a_A} = {} & (\ddot{x}_B \hat{I} + \ddot{y}_B \hat{J}) \\ & + (\ddot{x}_{A/B}\hat{\imath} + \ddot{y}_{A/B}\hat{\jmath}) + \vec{\omega} \times (\dot{x}_{A/B}\hat{\imath} + \dot{y}_{A/B}\hat{\jmath}) \\ & + \dot{\vec{\omega}} \times (x_{A/B}\hat{I} + y_{A/B}\hat{J}) \\ & + \vec{\omega} \times ((\dot{x}_{A/B}\hat{\imath} + \dot{y}_{A/B}\hat{\jmath}) + \vec{\omega} \times (x_{A/B}\hat{I} + y_{A/B}\hat{J}))\end{aligned}$$

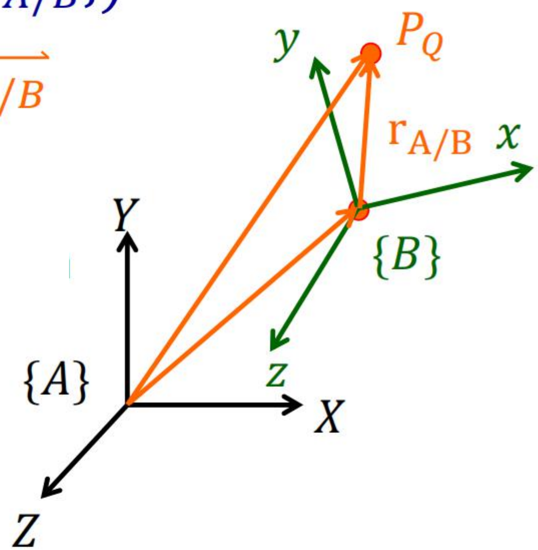


刚体的运动

$$\overrightarrow{a_A} = (\ddot{x}_B\hat{\imath} + \ddot{y}_B\hat{\jmath})$$
$$+\dot{\overrightarrow{\omega}} \times (x_{A/B}\hat{\imath} + y_{A/B}\hat{\jmath}) + \overrightarrow{\omega} \times (\overrightarrow{\omega} \times (x_{A/B}\hat{\imath} + y_{A/B}\hat{\jmath}))$$
$$+2\overrightarrow{\omega} \times (\dot{x}_{A/B}\hat{\imath} + \dot{y}_{A/B}\hat{\jmath}) + (\ddot{x}_{A/B}\hat{\imath} + \ddot{y}_{A/B}\hat{\jmath})$$

$$\Rightarrow \overrightarrow{a_A} = \overrightarrow{a_B} + \dot{\overrightarrow{\omega}} \times \overrightarrow{r_{A/B}} + \overrightarrow{\omega} \times \overrightarrow{\omega} \times \overrightarrow{r_{A/B}}$$
$$+\underbrace{2\overrightarrow{\omega} \times \overrightarrow{v_{rel}}} + \underbrace{\overrightarrow{a_{rel}}}$$

科氏加速度　　相对加速度



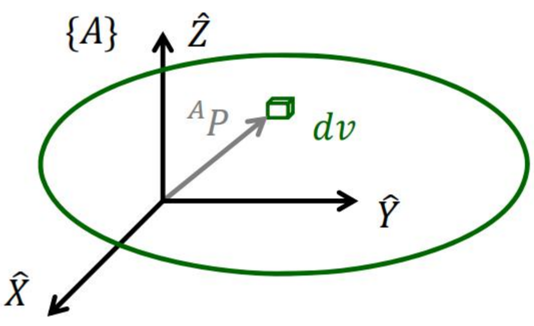
- 因此

$$^{A}A_Q = {}^{A}A_{B\,ORG}$$
$$+ {}^{A}\dot{\Omega}_B \times {}^{A}_{B}R\,{}^{B}P_Q + {}^{A}\Omega_B \times ({}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}P_Q)$$
$$+2\,{}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}_{B}R\,{}^{B}A_Q$$

质量分布

➢ 相对于坐标系 {A} 的惯性张量

$$^{A}I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}$$



惯性矩

$$I_{xx} = \iiint_V (y^2 + z^2)\rho dv$$

$$I_{yy} = \iiint_V (x^2 + z^2)\rho dv$$

$$I_{zz} = \iiint_V (x^2 + y^2)\rho dv$$

惯性积

$$I_{xy} = \iiint_V xy\rho dv$$

$$I_{xz} = \iiint_V xz\rho dv$$

$$I_{yz} = \iiint_V yz\rho dv$$

质量分布

➢ 惯性张量

◆实常数对称矩阵（可以正交对角化）

它的特征分解（也就是 $M = V\Lambda V^{-1}$）

$$ {}^{A}I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} = R \begin{bmatrix} I_{XX} & 0 & 0 \\ 0 & I_{YY} & 0 \\ 0 & 0 & I_{ZZ} \end{bmatrix} R^{T} $$

主惯性矩

旋转矩阵，揭示了主轴的方向

◆ $I_{xx} + I_{yy} + I_{zz} = trace({}^{A}I) = constant$

- 迹在相似变换下是不变的

◆如果xy平面是对称平面，则 $I_{xz} = I_{yz} = 0$

质量分布

- 平行轴原理
  - 计算惯性张量在参考坐标系平移下的变化

$$ {}^{A}I_{zz} = {}^{C}I_{zz} + m(x_c{}^2 + y_c{}^2) $$

$$ {}^{A}I_{xy} = {}^{C}I_{xy} - m x_c y_c $$

C：刚体的重心（COM）

A：任意坐标系

  - 矩阵向量形式

$$ {}^{A}I = {}^{C}I + m[P_c{}^T P_c I_3 - P_c P_c{}^T] $$

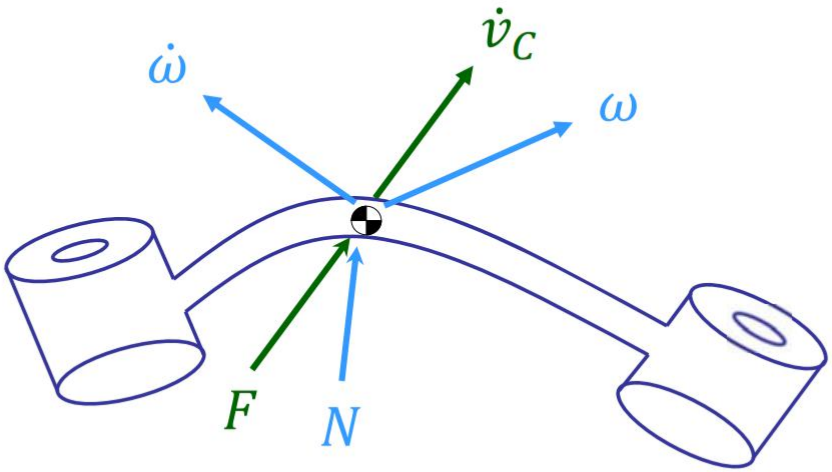
$$ P_c = [x_c \quad y_c \quad z_c]^T $$

相对于坐标系{A}的COM

牛顿方程和欧拉方程

- 牛顿方程

$$F = \frac{d}{dt}(m v_C) = m \dot{v}_C$$



- 欧拉方程

$$N = \frac{d}{dt}(I\omega)$$

即使使用惯性坐标系，它在运动过程中也可能变化

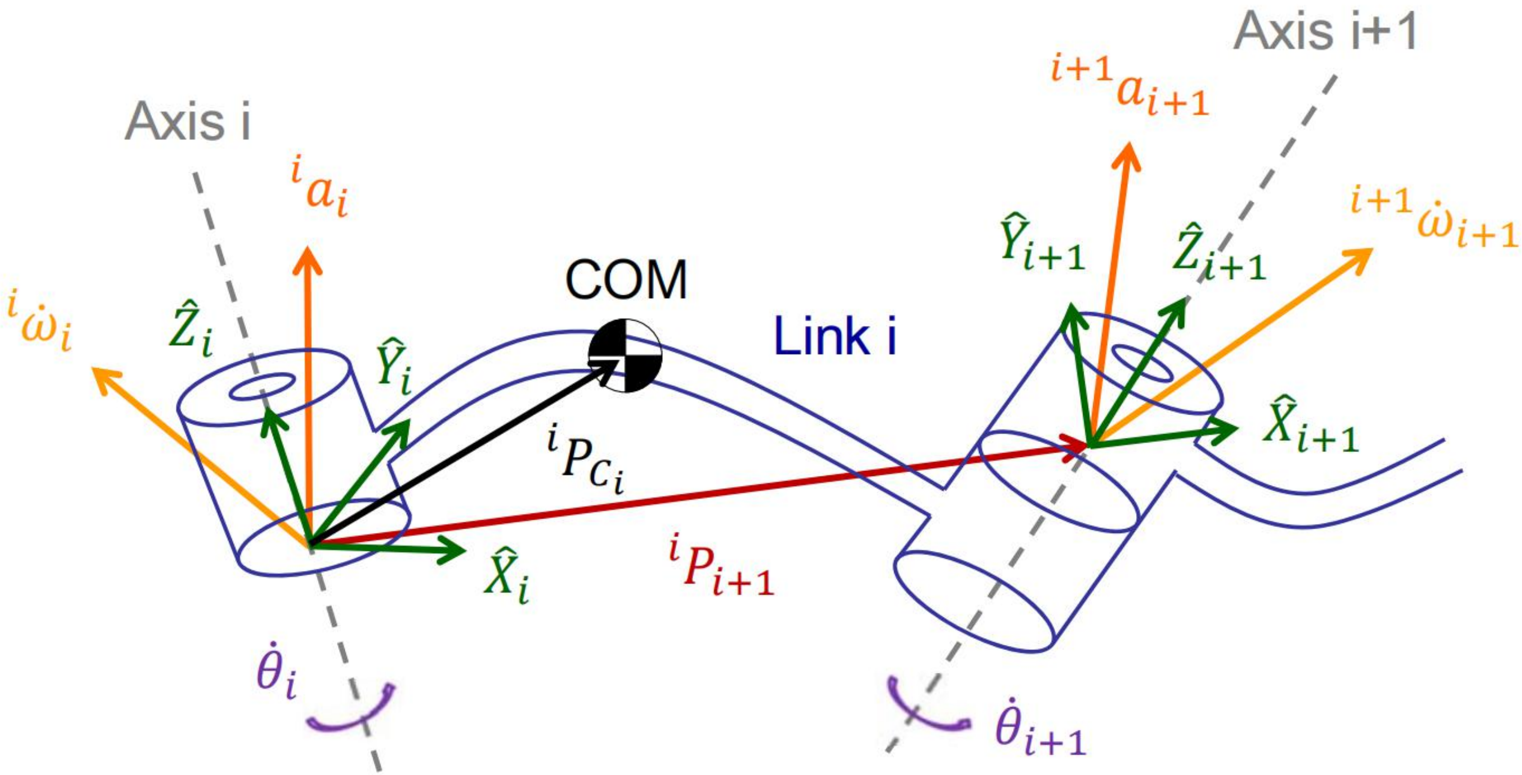
$$N = {}^{C}I\dot{\omega} + \omega \times {}^{C}I\omega$$

C：刚体坐标系，原点位于COM

${}^{C}I$：常数矩阵

加速度在连杆之间的传递

- 策略：表示坐标系{i}中连杆i的线加速度和角加速度，并找出它们与相邻连杆的线加速度和角加速度的关系



加速度在连杆之间的传递

- 旋转关节（连杆i+1）
  - 角加速度的传递

在第5章中

$$ {}^{i}\omega_{i+1} = {}^{i}\omega_{i} + {}^{i}_{i+1}R\dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} $$

↓ diff.

$$ \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} = {}^{i+1}\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix} $$

$$ {}^{i}\dot{\omega}_{i+1} = {}^{i}\dot{\omega}_{i} + {}^{i}\omega_{i} \times {}^{i}_{i+1}R\dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + {}^{i}_{i+1}R\ddot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} $$

↓ ${}^{i+1}_{i}R$

$$ {}^{i+i}\dot{\omega}_{i+1} = {}^{i+1}_{i}R\,{}^{i}\dot{\omega}_{i} + {}^{i+1}_{i}R\,{}^{i}\omega_{i} \times \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + \ddot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} $$

加速度在连杆之间的传递

◆ 线加速度的传递

$$^{i}a_{i+1} = {}^{i}a_i + {}^{i}\dot{\omega}_i \times {}^{i}P_{i+1} + {}^{i}\omega_i \times ({}^{i}\omega_i \times {}^{i}P_{i+1})$$

$$\Big\downarrow {}^{i+1}_{i}R$$

$$^{i+1}a_{i+1} = {}^{i+1}_{i}R({}^{i}a_i + {}^{i}\dot{\omega}_i \times {}^{i}P_{i+1} + {}^{i}\omega_i \times ({}^{i}\omega_i \times {}^{i}P_{i+1}))$$

加速度在连杆之间的传递

➢ 平动关节（连杆i+1）

◆ 角加速度的传递

$$ {}^{i}\dot{\omega}_{i+1} = {}^{i}\dot{\omega}_{i} \xrightarrow{{}^{i+1}_{i}R} {}^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{i}R\ {}^{i}\dot{\omega}_{i} $$

◆ 线加速度的传递

$$ {}^{i}a_{i+1} = {}^{i}a_{i} + {}^{i}\dot{\omega}_{i} \times {}^{i}P_{i+1} + {}^{i}\omega_{i} \times ({}^{i}\omega_{i} \times {}^{i}P_{i+1}) + 2\,{}^{i}\omega_{i} \times {}^{i}_{i+1}R\,\dot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + {}^{i}_{i+1}R\,\ddot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} $$

$$ \downarrow {}^{i+1}_{i}R $$

$$ \ddot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} = {}^{i+1}\begin{bmatrix} 0 \\ 0 \\ \ddot{d}_{i+1} \end{bmatrix} $$

$$ {}^{i+1}a_{i+1} = {}^{i+1}_{i}R\left({}^{i}a_{i} + {}^{i}\dot{\omega}_{i} \times {}^{i}P_{i+1} + {}^{i}\omega_{i} \times ({}^{i}\omega_{i} \times {}^{i}P_{i+1})\right) + 2\,{}^{i+1}\omega_{i+1} \times \dot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + \ddot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} $$

加速度在连杆之间的传递

- COM

$$ {}^{i}a_{C_i} = {}^{i}a_i + {}^{i}\dot{\omega}_i \times {}^{i}P_{C_i} + {}^{i}\omega_i \times ({}^{i}\omega_i \times {}^{i}P_{C_i}) $$

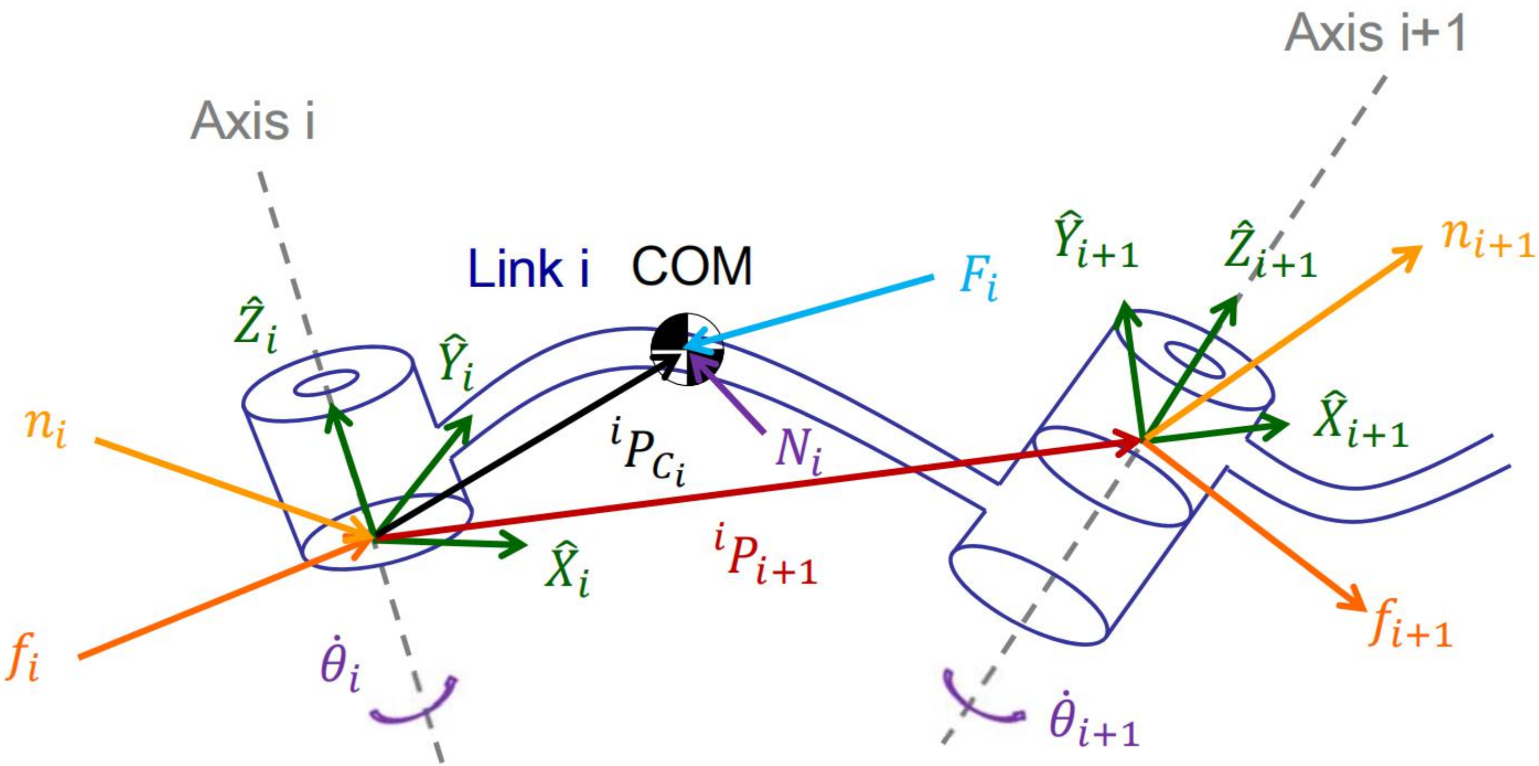
C_i: 第i个连杆的COM

力在连杆之间的传递

➢ 作用在COM上的惯性力和力矩

$$F_i = ma_{C_i}$$

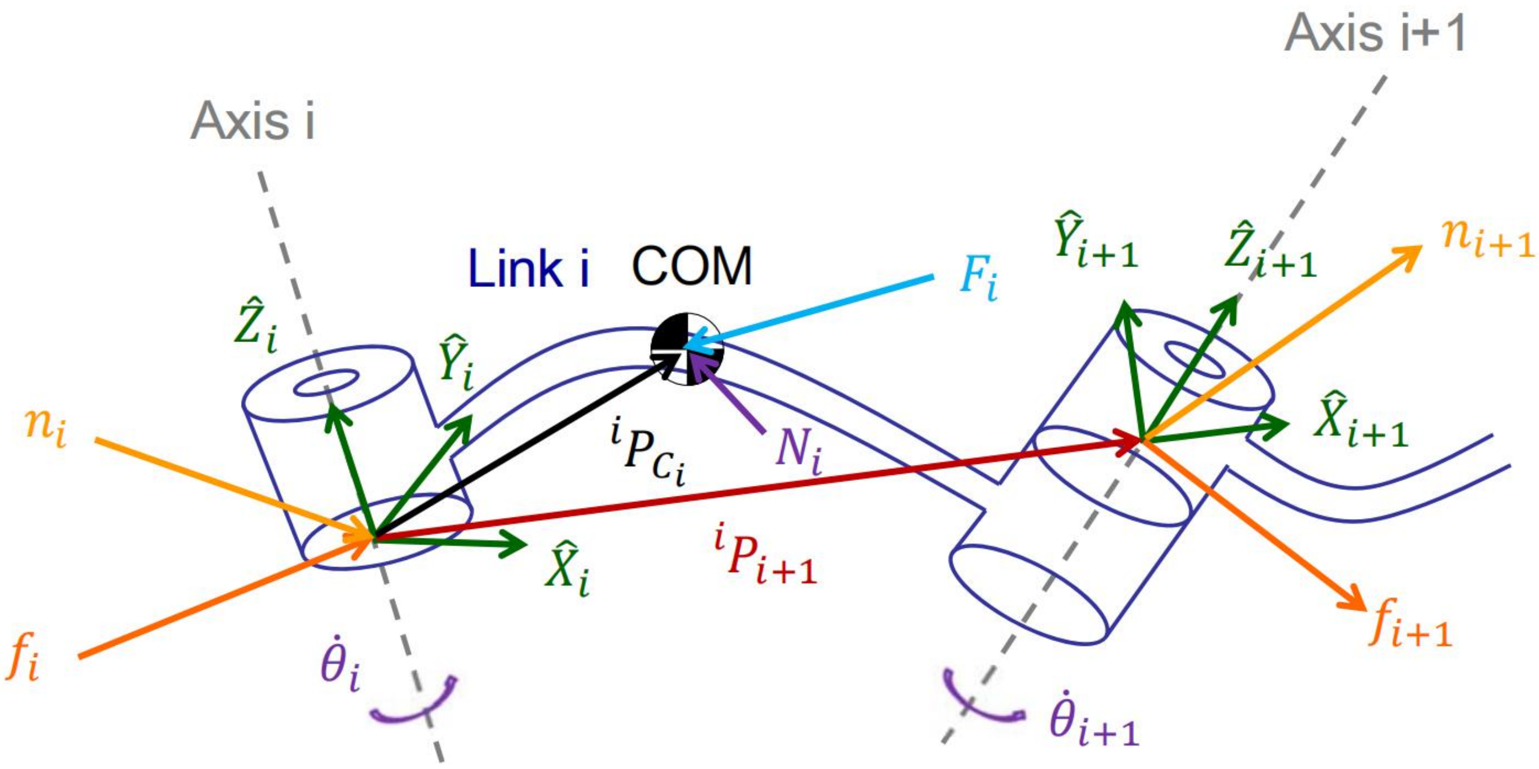
$$N_i = {}^{C_i}I\dot{\omega}_i + \omega_i \times {}^{C_i}I\omega_i$$



力在连杆之间的传递

$$ {}^{i}f_i = {}^{i}_{i+1}R\, {}^{i+1}f_{i+1} + {}^{i}F_i $$

$$ {}^{i}n_i = {}^{i}_{i+1}R\, {}^{i+1}n_{i+1} + {}^{i}N_i + {}^{i}P_{C_i} \times {}^{i}F_i + {}^{i}P_{i+1} \times {}^{i}_{i+1}R\, {}^{i+1}f_{i+1} $$



力在连杆之间的传递

- 因此
  - 旋转关节

$$\tau_i = {}^{i}n_i^T \, {}^{i}\hat{Z}_i$$

  - 平动关节

$$\tau_i = {}^{i}f_i^T \, {}^{i}\hat{Z}_i$$

- 结论
  - 包括重力： ${}^{0}a_0 = g = 9.81 \, m/s$
  - 机械臂在自由空间移动 ${}^{N+1}f_{N+1} = 0$ ${}^{N+1}n_{N+1} = 0$

迭代牛顿欧拉动力学方程

- 向外迭代
  - 连杆1到连杆n
  - 速度和加速度
- 向内迭代
  - 连杆n到连杆1:
  - 力和力矩
- 旋转关节和平动关节：选择正确的方程
- 通用结构，可以应用于任何机械臂
- 方便数字计算

例：一个RR机械臂

- 条件

$$^{1}P_{C_1} = l_1\hat{X}_1 \qquad ^{C_1}I_1 = 0$$

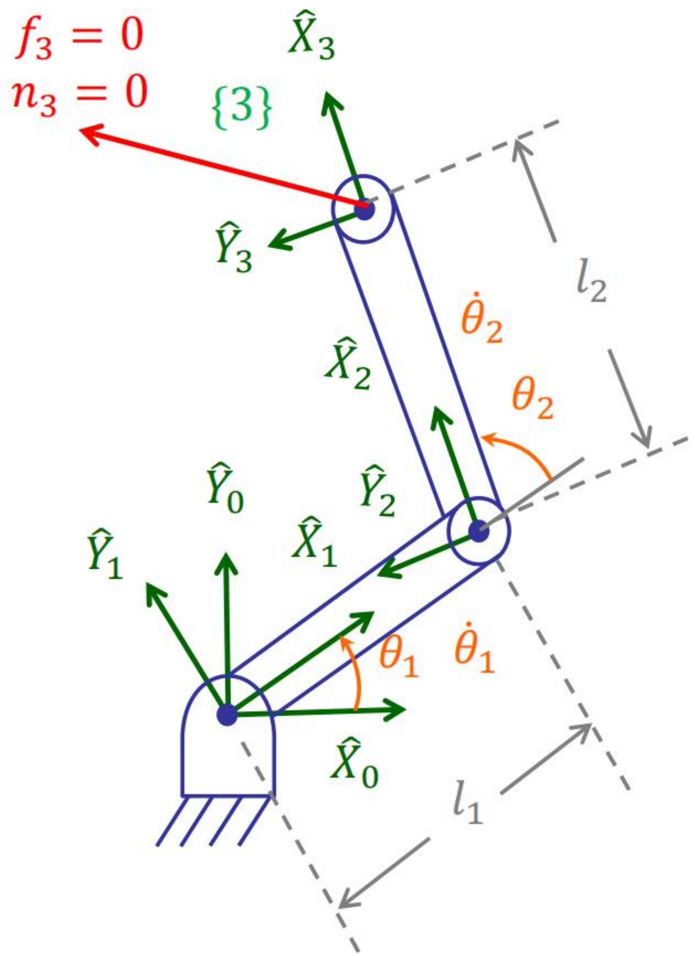
$$^{2}P_{C_2} = l_2\hat{X}_2 \qquad ^{C_2}I_2 = 0$$

$$m_1, m_2$$

$$\omega_0 = 0 \qquad ^{0}\dot{v}_0 = g\hat{Y}_0$$

$$\dot{\omega}_0 = 0$$

$$^{i}_{i+1}R = \begin{bmatrix} c_{i+1} & -s_{i+1} & 0 \\ s_{i+1} & c_{i+1} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



例：一个RR机械臂

➢ 速度和加速度的传递

$$^{1}\omega_1 = {}^{1}_{0}R\ {}^{0}\omega_0 + \dot{\theta}_1\ {}^{1}\hat{Z}_1 = \dot{\theta}_1\ {}^{1}\hat{Z}_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}$$

$$^{1}\dot{\omega}_1 = {}^{1}_{0}R\ {}^{0}\dot{\omega}_0 + {}^{1}_{0}R\ {}^{0}\omega_0 \times \dot{\theta}_1\ {}^{1}\hat{Z}_1 + \ddot{\theta}_1 \hat{Z}_1 = \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{bmatrix}$$

$$^{1}a_1 = {}^{1}_{0}R\left({}^{0}a_0 + {}^{0}\omega_0 \times {}^{0}P_1 + {}^{0}\omega_0 \times \left({}^{0}\omega_0 \times {}^{0}P_1\right)\right) = \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ g \\ 0 \end{bmatrix} = \begin{bmatrix} gs_1 \\ gc_1 \\ 0 \end{bmatrix}$$

$$^{1}a_{C_1} = {}^{1}a_1 + {}^{1}\dot{\omega}_1 \times {}^{1}P_{C_1} + {}^{1}\omega_1 \times \left({}^{1}\omega_1 \times {}^{1}P_{C_1}\right)$$

$$= \begin{bmatrix} gs_1 \\ gc_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ l_1\ddot{\theta}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} -l_1\dot{\theta}_1^{\ 2} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -l_1\dot{\theta}_1^{\ 2} + gs_1 \\ l_1\ddot{\theta}_1 + gc_1 \\ 0 \end{bmatrix}$$

例：一个RR机械臂

$$
{}^1F_1 = m\,{}^1a_{C_1} \begin{bmatrix} -m_1 l_1 \dot{\theta}_1^{\,2} + m_1 g s_1 \\ m_1 l_1 \ddot{\theta}_1 + m_1 g c_1 \\ 0 \end{bmatrix}
$$

$$
{}^1N_1 = {}^{C_1}I\,{}^1\dot{\omega}_1 + {}^1\omega_1 \times {}^{C_1}I\,{}^1\omega_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
{}^2\omega_2 = {}^2_1R\,{}^1\omega_1 + \dot{\theta}_2\,{}^2\hat{Z}_2 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix}
$$

$$
{}^2\dot{\omega}_2 = {}^2_1R\,{}^1\dot{\omega}_1 + {}^2_1R\,{}^1\omega_1 \times \dot{\theta}_2\,{}^2\hat{Z}_2 + \ddot{\theta}_2\,{}^2\hat{Z}_2 = \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 + \ddot{\theta}_2 \end{bmatrix}
$$

$$
{}^2a_2 = {}^2_1R\left({}^1a_1 + {}^1\dot{\omega}_1 \times {}^1P_2 + {}^1\omega_1 \times \left({}^1\omega_1 \times {}^1P_2\right)\right) =
$$

$$
= \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -l_1 \dot{\theta}_1^{\,2} + g s_1 \\ l_1 \ddot{\theta}_1 + g c_1 \\ 0 \end{bmatrix} = \begin{bmatrix} l_1 \ddot{\theta}_1 s_2 - l_1 \dot{\theta}_1^{\,2} c_2 + g s_{12} \\ l_1 \ddot{\theta}_1 c_2 + l_1 \dot{\theta}_1^{\,2} s_2 + g c_{12} \\ 0 \end{bmatrix}
$$

例：一个RR机械臂

$$
{}^2a_{C_2} = {}^2a_2 + {}^2\dot{\omega}_2 \times {}^2P_{C_2} + {}^2\omega_2 \times \left({}^2\omega_2 \times {}^2P_{C_2} \right)
$$

$$
= \begin{bmatrix} l_1\ddot{\theta}_1 s_2 - l_1 \dot{\theta}_1^{\,2} c_2 + g s_{12} \\ l_1\ddot{\theta}_1 c_2 + l_1 \dot{\theta}_1^{\,2} s_2 + g c_{12} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ l_2(\ddot{\theta}_1 + \ddot{\theta}_2) \\ 0 \end{bmatrix} + \begin{bmatrix} -l_2(\dot{\theta}_1 + \dot{\theta}_2)^2 \\ 0 \\ 0 \end{bmatrix}
$$

$$
{}^2F_2 = m\, {}^2a_{C_2} = \begin{bmatrix} m_2 l_1\ddot{\theta}_1 s_2 - m_2 l_1\dot{\theta}_1^{\,2} c_2 + m_2 g s_{12} - m_2 l_2(\dot{\theta}_1 + \dot{\theta}_2)^2 \\ m_2 l_1\ddot{\theta}_1 c_2 + m_2 l_1\dot{\theta}_1^{\,2} s_2 + m_2 g c_{12} + m_2 l_2(\ddot{\theta}_1 + \ddot{\theta}_2) \\ 0 \end{bmatrix}
$$

$$
{}^2N_2 = {}^{C_2}I\, {}^2\dot{\omega}_2 + {}^2\omega_2 \times {}^{C_2}I\, {}^2\omega_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

例：一个RR机械臂

- 力和力矩的传递

$$^2f_2 = {}^2_3R\,{}^3f_3 + {}^2F_2 = {}^2F_2$$

$$^2n_2 = {}^2_3R\,{}^3n_3 + {}^2N_2 + {}^2P_{C_2} \times {}^2F_2 + {}^2P_3 \times {}^2_3R\,{}^3f_3 = \begin{bmatrix} 0 \\ 0 \\ m_2 l_1 l_2 c_2 \ddot{\theta}_1 + m_2 l_1 l_2 s_2 \dot{\theta}_1^{\,2} + m_2 l_2 g c_{12} + m_2 l_2^{\,2}(\ddot{\theta}_1 + \ddot{\theta}_2) \end{bmatrix}$$

$$^1f_1 = {}^1_2R\,{}^2f_2 + {}^1F_1$$

$$= \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} m_2 l_1 s_2 \ddot{\theta}_1 - m_2 l_1 c_2 \dot{\theta}_1^{\,2} + m_2 g s_{12} - m_2 l_2 (\dot{\theta}_1 + \dot{\theta}_2)^2 \\ m_2 l_1 c_2 \ddot{\theta}_1 + m_2 l_1 s_2 \dot{\theta}_1^{\,2} + m_2 g c_{12} + m_2 l_2 (\ddot{\theta}_1 + \ddot{\theta}_2) \\ 0 \end{bmatrix}$$

$$+ \begin{bmatrix} -m_1 l_1 \dot{\theta}_1^{\,2} + m_1 g s_1 \\ m_1 l_1 \ddot{\theta}_1 + m_1 g c_1 \\ 0 \end{bmatrix}$$

例：一个RR机械臂

$$
\begin{aligned}
{}^{1}n_1 &= {}^{1}_{2}R\ {}^{2}n_2 + {}^{1}N_1 + {}^{1}P_{C_1} \times {}^{1}F_1 + {}^{1}P_2 \times {}^{1}_{2}R\ {}^{2}f_2 \\
&= \begin{bmatrix} 0 \\ 0 \\ m_2 l_1 l_2 c_2 \ddot{\theta}_1 + m_2 l_1 l_2 s_2 \dot{\theta}_1{}^2 + m_2 l_2 g c_{12} + m_2 l_2{}^2 (\ddot{\theta}_1 + \ddot{\theta}_2) \end{bmatrix} \\
&\quad + \begin{bmatrix} 0 \\ 0 \\ m_1 l_1{}^2 \ddot{\theta}_1 + m_1 l_1 g c_1 \end{bmatrix} \\
&\quad + \begin{bmatrix} 0 \\ 0 \\ m_2 l_1{}^2 \ddot{\theta}_1 - m_2 l_1 l_2 s_2 (\dot{\theta}_1 + \dot{\theta}_2)^2 + m_2 l_1 g s_2 s_{12} + m_2 l_1 l_2 c_2 (\ddot{\theta}_1 + \ddot{\theta}_2) + m_2 l_1 g c_2 c_{12} \end{bmatrix}
\end{aligned}
$$

例：一个RR机械臂

- 关节力矩

$$\tau_1 = {}^1n_1^T \, {}^1\widehat{Z_1}$$
$$= m_2 l_2{}^2(\ddot{\theta}_1 + \ddot{\theta}_2) + m_2 l_1 l_2 c_2 (2\ddot{\theta}_1 + \ddot{\theta}_2) + (\mathrm{m}_1 + \mathrm{m}_2) l_1{}^2 \ddot{\theta}_1$$
$$- m_2 l_1 l_2 s_2 \dot{\theta}_2{}^2 - 2 m_2 l_1 l_2 s_2 \dot{\theta}_1 \dot{\theta}_2 + m_2 l_2 g c_{12} + (m_1 + m_2) l_1 g c_1$$

$$\tau_2 = {}^2n_2^T \, {}^2\widehat{Z_2}$$
$$= m_2 l_1 l_2 c_2 \ddot{\theta}_1 + m_2 l_1 l_2 s_2 \dot{\theta}_1{}^2 + m_2 l_2 g c_{12} + m_2 l_2{}^2 (\ddot{\theta}_1 + \ddot{\theta}_2)$$

动力学方程的结构

- 状态空间方程

$$\tau = M(\Theta)\ddot{\Theta} + V(\Theta,\dot{\Theta}) + G(\Theta)$$

$n\times 1$ $n\times n$ $n\times 1$ $n\times 1$ $n\times 1$

质量矩阵　离心力和科氏力矩阵　重力矩阵

- 回顾RR机械臂

$$M(\Theta) = \begin{bmatrix} {l_2}^2 m_2 + 2l_1 l_2 m_2 c_2 + {l_1}^2(m_1+m_2) & {l_2}^2 m_2 + l_1 l_2 m_2 c_2 \\ {l_2}^2 m_2 + l_1 l_2 m_2 c_2 & {l_2}^2 m_2 \end{bmatrix}$$

$$V(\Theta,\dot{\Theta}) = \begin{bmatrix} -m_2 l_1 l_2 s_2 {\dot{\theta}_2}^2 - 2m_2 l_1 l_2 s_2 \dot{\theta}_1 \dot{\theta}_2 \\ m_2 l_1 l_2 s_2 {\dot{\theta}_1}^2 \end{bmatrix}$$

$$G(\Theta) = \begin{bmatrix} m_2 l_2 g c_{12} + (m_1+m_2) l_1 g c_1 \\ m_2 l_2 g c_{12} \end{bmatrix}$$

动力学方程的结构

- 位形空间方程

$$\tau = M(\Theta)\ddot{\Theta} + B(\Theta)[\dot{\Theta}\dot{\Theta}] + C(\Theta)[\dot{\Theta}^2] + G(\Theta)$$

$n \times 1$ $n \times n$ 质量矩阵 $n \times \frac{n(n-1)}{2}$ 科氏力矩阵 $n \times 1$ $n \times n$ 离心力矩阵 $n \times 1$ 重力矩阵

$$[\dot{\Theta}\dot{\Theta}] = [\dot{\theta}_1\dot{\theta}_2 \quad \dot{\theta}_1\dot{\theta}_3 \quad \dots \quad \dot{\theta}_{n-1}\dot{\theta}_n]^T$$

$\frac{n(n-1)}{2} \times 1$

$$[\dot{\Theta}^2] = [\dot{\theta}_1{}^2 \quad \dot{\theta}_2{}^2 \quad \dots \quad \dot{\theta}_n{}^2]^T$$

$n \times 1$

动力学方程的结构

- 回顾RR机械臂

$$V(\Theta,\dot{\Theta})=\begin{bmatrix}-m_2l_1l_2s_2\dot{\theta}_2{}^2-2m_2l_1l_2s_2\dot{\theta}_1\dot{\theta}_2\\ m_2l_1l_2s_2\dot{\theta}_1{}^2\end{bmatrix}=B(\Theta)[\dot{\Theta}\dot{\Theta}]+C(\Theta)[\dot{\Theta}^2]$$

$$B(\Theta)=\begin{bmatrix}-2m_2l_1l_2s_2\\ 0\end{bmatrix}\qquad [\dot{\Theta}\dot{\Theta}]=[\dot{\theta}_1\dot{\theta}_2]$$

$$C(\Theta)=\begin{bmatrix}0 & -m_2l_1l_2s_2\\ m_2l_1l_2s_2 & 0\end{bmatrix}\qquad [\dot{\Theta}^2]=\begin{bmatrix}\dot{\theta}_1{}^2\\ \dot{\theta}_2{}^2\end{bmatrix}$$

机械臂动力学的拉格朗日方程

- 牛顿-欧拉法：基于力和力矩的分析
- 拉格朗日：基于能量的分析

- 对于一个系统来说，两种方法应该得到同样的运动方程

机械臂动力学的拉格朗日方程

➢ 动能

$$k_i = \frac{1}{2} m_i v_{c_i}^T v_{c_i} + \frac{1}{2} \, {}^i\omega_i^T \, {}^{C_i}I_i \, {}^i\omega_i$$

$$k = \sum_{i=1}^{n} k_i \qquad\qquad k = k(\Theta, \dot{\Theta}) = \frac{1}{2} \dot{\Theta}^T M(\Theta) \dot{\Theta}$$

➢ 势能（位能）

$$u_i = -m_i \, {}^0g^T \, {}^0P_{C_i} + \underline{u_{ref_i}}$$

Shift the zero reference height

$$u = \sum_{i=1}^{n} u_i \qquad\qquad u = u(\Theta)$$

机械臂动力学的拉格朗日方程

- 拉格朗日

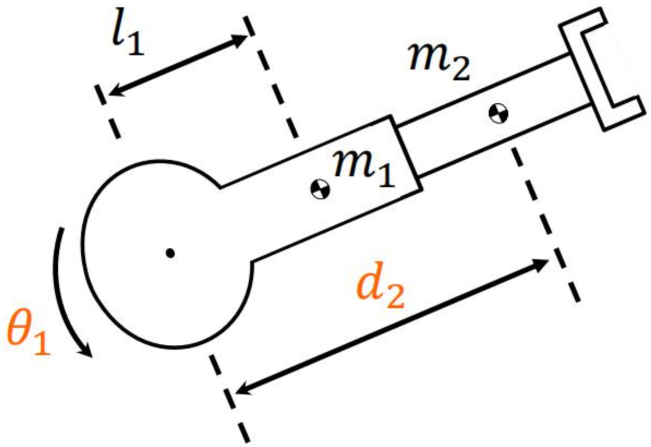
$$\mathcal{L}(\Theta, \dot{\Theta}) = k(\Theta, \dot{\Theta}) - u(\Theta)$$

- 机械臂的运动方程

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\Theta}} - \frac{\partial \mathcal{L}}{\partial \Theta} = \tau$$

$$\frac{d}{dt}\frac{\partial k}{\partial \dot{\Theta}} - \frac{\partial k}{\partial \Theta} + \frac{\partial u}{\partial \Theta} = \tau$$

例：一个RP机械臂



$$^{C_1}I_1 = \begin{bmatrix} I_{xx1} & 0 & 0 \\ 0 & I_{yy1} & 0 \\ 0 & 0 & I_{zz1} \end{bmatrix}$$

$$^{C_2}I_2 = \begin{bmatrix} I_{xx2} & 0 & 0 \\ 0 & I_{yy2} & 0 \\ 0 & 0 & I_{zz2} \end{bmatrix}$$

➢ 动能

$$k_1 = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} I_{zz1} \dot{\theta}_1^2$$

$$k_2 = \frac{1}{2} m_2 (d_2^2 \dot{\theta}_1^2 + \dot{d}_2^2) + \frac{1}{2} I_{zz2} \dot{\theta}_1^2$$

$$k(\Theta, \dot{\Theta}) = \frac{1}{2} (m_1 l_1^2 + I_{zz1} + I_{zz2} + m_2 d_2^2) \dot{\theta}_1^2 + \frac{1}{2} m_2 \dot{d}_2^2$$

例：一个RP机械臂

- 势能

$$u_1 = m_1 g l_1 sin\theta_1 + m_1 g l_1$$

$$u_2 = m_2 g d_2 sin\theta_1 + m_2 g d_{2max}$$

$$u(\Theta) = (m_1 l_1 + m_2 d_2) g sin\theta_1 + \underline{m_1 g l_1 + m_2 g d_{2max}}$$

Shift the zero reference height

- 拉格朗日

$$\frac{\partial k}{\partial \dot{\Theta}} = \begin{bmatrix} (m_1 l_1^2 + I_{zz1} + I_{zz2} + m_2 d_2^2)\dot{\theta}_1 \\ m_2 \dot{d}_2 \end{bmatrix}$$

$$\frac{\partial k}{\partial \Theta} = \begin{bmatrix} 0 \\ m_2 d_2 \ \dot{\theta}_1^2 \end{bmatrix}$$

$$\frac{\partial u}{\partial \Theta} = \begin{bmatrix} (m_1 l_1 + m_2 d_2) g cos\theta_1 \\ m_2 g sin\,\theta_1 \end{bmatrix}$$

例：一个RP机械臂

- 运动方程

$$\tau_1 = (m_1 l_1^2 + I_{zz1} + I_{zz2} + m_2 d_2^2)\ddot{\theta}_1 + 2m_2 d_2 \dot{\theta}_1 \dot{d}_2 + (m_1 l_1 + m_2 d_2) g cos\theta_1$$

$$\tau_2 = m_2 \ddot{d}_2 - m_2 d_2 \dot{\theta}_1^2 + m_2 g sin\theta_1$$

- 状态空间表达

$$\tau = M(\Theta)\ddot{\Theta} + V(\Theta, \dot{\Theta}) + G(\Theta)$$

$$M(\Theta) = \begin{bmatrix} m_1 l_1^2 + I_{zz1} + I_{zz2} + m_2 d_2^2 & 0 \\ 0 & m_2 \end{bmatrix}$$

$$V(\Theta, \dot{\Theta}) = \begin{bmatrix} 2m_2 d_2 \dot{\theta}_1 \dot{d}_2 \\ -m_2 d_2 \dot{\theta}_1^2 \end{bmatrix}$$

$$G(\Theta) = \begin{bmatrix} (m_1 l_1 + m_2 d_2) g cos\theta_1 \\ m_2 g sin\theta_1 \end{bmatrix}$$

笛卡尔空间中的机械臂动力学

- 动态方程
  - 关节空间

$$\tau = M(\Theta)\ddot{\Theta} + V(\Theta,\dot{\Theta}) + G(\Theta)$$

  - 笛卡尔空间

$$F = M_x(\Theta)\ddot{X} + V_x(\Theta,\dot{\Theta}) + G_x(\Theta)$$

- 方程

$$\tau = J^T(\Theta)F$$

$$F = J^{-T}\tau = J^{-T}M(\Theta)\ddot{\Theta} + J^{-T}V(\Theta,\dot{\Theta}) + J^{-T}G(\Theta)$$

$$\dot{X} = J\dot{\Theta} \qquad \ddot{X} = \dot{J}\dot{\Theta} + J\ddot{\Theta} \qquad \ddot{\Theta} = J^{-1}\ddot{X} - J^{-1}\dot{J}\dot{\Theta}$$

笛卡尔空间中的机械臂动力学

$$F = J^{-T}M(\Theta)J^{-1}\ddot{X} - J^{-T}M(\Theta)J^{-1}\dot{J}\dot{\Theta} + J^{-T}V(\Theta,\dot{\Theta}) + J^{-T}G(\Theta)$$
$$= M_x(\Theta)\ddot{X} + V_x(\Theta,\dot{\Theta}) + G_x(\Theta)$$

$$M_x(\Theta) = J^{-T}(\Theta)M(\Theta)J^{-1}(\Theta)$$

$$V_x(\Theta,\dot{\Theta}) = J^{-T}(\Theta)\left(V(\Theta,\dot{\Theta}) - M(\Theta)J^{-1}(\Theta)\dot{J}(\Theta)\dot{\Theta}\right)$$

$$G_x(\Theta) = J^{-T}(\Theta)G(\Theta)$$

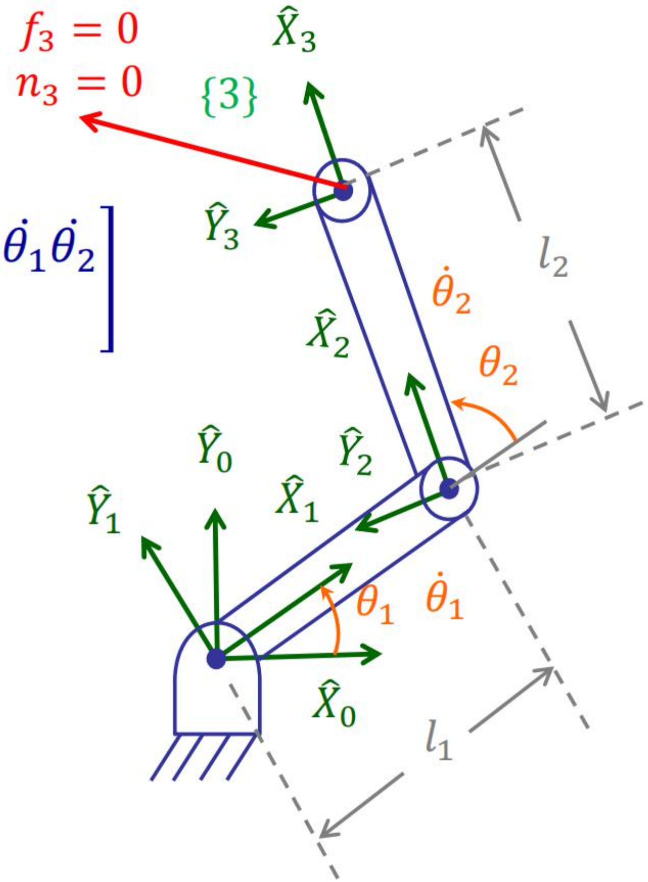
回顾例：一个RP机械臂

- 关节空间

$$M(\Theta)=\begin{bmatrix} l_2{}^2 m_2+2l_1l_2m_2c_2+l_1{}^2(m_1+m_2) & l_2{}^2m_2+l_1l_2m_2c_2 \\ l_2{}^2m_2+l_1l_2m_2c_2 & l_2{}^2m_2 \end{bmatrix}$$

$$V(\Theta,\dot{\Theta})=\begin{bmatrix} -m_2l_1l_2s_2\dot{\theta}_2{}^2-2m_2l_1l_2s_2\dot{\theta}_1\dot{\theta}_2 \\ m_2l_1l_2s_2\dot{\theta}_1{}^2 \end{bmatrix}$$

$$G(\Theta)=\begin{bmatrix} m_2l_2gc_{12}+(m_1+m_2)l_1gc_1 \\ m_2l_2gc_{12} \end{bmatrix}$$



回顾例：一个RP机械臂

- 雅可比矩阵

$$J(\Theta)=\begin{bmatrix} l_1 s_2 & 0 \\ l_1 c_2 + l_2 & l_2 \end{bmatrix} \qquad J^{-1}(\Theta)=\frac{1}{l_1 l_2 s_2}\begin{bmatrix} l_2 & 0 \\ -l_1 c_2 - l_2 & l_1 s_2 \end{bmatrix}$$

$$\dot{J}(\Theta)=\begin{bmatrix} l_1 c_2 \dot{\theta}_2 & 0 \\ -l_1 s_2 \theta_2 & 0 \end{bmatrix}$$

- 笛卡尔空间

$$M_x(\Theta)=J^{-T}(\Theta)M(\Theta)J^{-1}(\Theta)=\begin{bmatrix} m_2+\frac{m_1}{{s_2}^2} & 0 \\ 0 & m_2 \end{bmatrix}$$

$$V_x\left(\Theta,\dot{\Theta}\right)=J^{-T}(\Theta)\left(V\left(\Theta,\dot{\Theta}\right)-M(\Theta)J^{-1}(\Theta)\dot{J}(\Theta)\dot{\Theta}\right)$$

$$=\begin{bmatrix} -(m_2 l_1 c_2 + m_2 l_2){\dot{\theta}_1}^2 - m_2 l_2 {\dot{\theta}_2}^2 - (2 m_2 l_2 + m_2 l_1 c_2 + m_1 l_1 \frac{c_2}{{s_2}^2})\dot{\theta}_1 \dot{\theta}_2 \\ m_2 l_1 s_2 {\dot{\theta}_1}^2 + m_2 l_1 s_2 \dot{\theta}_1 \dot{\theta}_2 \end{bmatrix}$$

$$G_x(\Theta)=J^{-T}(\Theta)G(\Theta)=\begin{bmatrix} m_1 g \frac{c_1}{s_2} + m_2 g s_{12} \\ m_2 g c_{12} \end{bmatrix}$$

力矩方程

➢ 笛卡尔空间

$$\tau = J^T(\Theta)F = J^T(\Theta)(M_x(\Theta)\ddot{X} + V_x(\Theta,\dot{\Theta}) + G_x(\Theta))$$

$$\tau = J^T(\Theta)M_x(\Theta)\ddot{X} + B_x(\Theta)[\dot{\Theta}\dot{\Theta}] + C_x(\Theta)[\dot{\theta}^2] + G(\Theta)$$

$$\Rightarrow \quad J^T(\Theta)V_x(\Theta,\dot{\Theta}) = B_x(\Theta)[\dot{\Theta}\dot{\Theta}] + C_x(\Theta)[\dot{\theta}^2]$$

回顾例：一个RP机械臂

$$J^T(\Theta)V_x(\Theta,\dot{\Theta}) = B_x(\Theta)[\dot{\Theta}\dot{\Theta}] + C_x(\Theta)[\dot{\theta}^2]$$

$$= \begin{bmatrix} l_1 s_2 & l_1 c_2 + l_2 \\ 0 & l_2 \end{bmatrix} \begin{bmatrix} -(m_2 l_1 c_2 + m_2 l_2)\dot{\theta}_1{}^2 - m_2 l_2 \dot{\theta}_2{}^2 - (2m_2 l_2 + m_2 l_1 c_2 + m_1 l_1 \frac{c_2}{s_2{}^2})\dot{\theta}_1\dot{\theta}_2 \\ m_2 l_1 s_2 \dot{\theta}_1{}^2 + l_1 m_2 s_2 \dot{\theta}_1 \dot{\theta}_2 \end{bmatrix}$$

$$B_x(\Theta) = \begin{bmatrix} -m_1 l_1{}^2 \frac{c_2}{s_2} - m_2 l_1 l_2 s_2 \\ m_2 l_1 l_2 s_2 \end{bmatrix}$$

$$C_x(\Theta) = \begin{bmatrix} 0 & -m_2 l_1 l_2 s_2 \\ m_2 l_1 l_2 s_2 & 0 \end{bmatrix}$$

摩擦力

- 粘性摩擦

$$\tau_{friction} = c\dot{\theta}$$

- 库伦摩擦

$$\tau_{friction} = c\, sgn\dot{\theta}$$

$\dot{\theta} = 0,\ c =$ “static coefficient”

$\dot{\theta} \neq 0,\ c =$ “dynamic coefficient”

谢 谢！